An LMI-based Analysis of an Adaptive Flight Control System with Unmatched Uncertainties

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In this paper we formulate the stability and the performance problem of a $\sigma-$ modification based adaptive controller in the presence of an unmatched uncertainty. The formulation involves casting the error dynamics of an adaptive control system into a Linear Parameter Varying form that was initiated in the authors' previous work. The analysis framework takes the viewpoint that nonlinear systems, in particular some classes of adaptive controllers, can be parameterized by a set of linear systems, and linear matrix inequaility tools can be utilized for the analysis of the robustness of adaptive controllers to a class of unmatched uncertainties that violate standard assumptions employed in adaptive control.

I. Introduction

In this paper, we address the application of a Linear Matrix Inequality (LMI)-based tool for analyzing the stability characteristics and the performance degradation of an adaptive flight control system in the presence of unmatched uncertainties. Standard adaptive control methods[1–6] employ time-varying parameters that do not fit the traditional stability and robustness validation process for flight control systems, which is mandatory for flight certification. The incorporation of time-varying adaptation laws fundamentally changes the characterization of stability from exponential stability to weaker assurances such of either asymptotic stability or uniform ultimate boundedness (UUBness) of the tracking errors,[3] which has been a fundamental obstacle in ensuring robustness of adaptive control. The traditional validation procedure is based on linearized dynamics around a trim point, which has been justified by Lyapunov's first theorem, and required a closed-loop system to be exponentially stable. In adaptive control, this can only be attained under highly restrictive persistency of excitation condition. Consequently, it can not be claimed that adaptive control is robust to unmatched uncertainties, unmodeled dynamics, external disturbances.[7]

Employing modification terms in adaptive laws, such as $\sigma-$modification,[8] $e-$modification,[1] and projection,[9] weakens the notion of stability from asymptotic convergence to UUBness of tracking errors.[10] This notion is useful for neural network (NN)-based adaptive algorithms[10–12] because the

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inherent network approximation error necessitates the employment of modifications for proof of boundedness for closed-loop signals. In the case of a NN-based adaptive algorithm, this has been the price that one has to pay in order to eliminate the requirement of having a perfectly known regression vector. Nevertheless, even though incorporating modification terms in adaptive laws has shown to be robust to a limited class of modeling errors that violates the standard assumptions in adaptive control,[3] a general framework to quantitatively analyze the performance of adaptive control systems with respect to unstructured uncertainties and unmodeled dynamics has remained a very challenging problem.

In Ref.s 13,14 we have initiated an effort that addresses quantifying nominal system performance and stability margins of adaptive flight control using an LMI framework. This paper is an extension that includes analysis of an adaptive control system in the presence of unmatched uncertainties. In other words, following Ref.s 13,14, the combined error dynamics, composed of the tracking error and the weight estimate error, are cast as an exponentially stable system under bounded perturbation by employing $\sigma-$modification as an essential ingredient. The exponentially perturbed system is then viewed as a linear-parameter varying (LPV) system, and LMI analysis tools are applied for the analysis of the performance and the robustness in the presence of unmatched uncertainties.

The paper is organized as follows. In Section II, we introduce an affine parametrization and formulate the analysis problems. In Section III, we show that qunatifying the $\mathcal{L}_2$ gain of an unmatched uncertainty essentially falls into a robust stability problem. In Section IV, the performance analysis for the tracking error is carried by restricting the unmatched uncertainty to a static mapping, which reveals that the formulation falls into robust a performance problem. Simulation results are presented in Section V, which is followed by concluding remarks in Section VI.

II. Problem Formulation

Consider a single-input single-output system described by:

$$\begin{aligned}\dot{\boldsymbol{x}}(t) =& A\boldsymbol{x}(t) + \boldsymbol{b}(u(t) + \boldsymbol{W}^\top \boldsymbol{\phi}(\boldsymbol{x}(t))) + B_u \boldsymbol{\Delta}_u(\boldsymbol{z}(t)),\\ \boldsymbol{z}(t) =& C_u \boldsymbol{x}(t),\\ y(t) =& \boldsymbol{c}^\top \boldsymbol{x}(t),\end{aligned} \tag{1}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$ is the system state vector, $u(t) \in \mathbb{R}$ is the input, $y(t) \in \mathbb{R}$ is the output, $\boldsymbol{W} \in \mathbb{R}^N$ is a uncertain parameter vector, $\boldsymbol{\phi}(\boldsymbol{x}(t)) \in \mathbb{R}^N$ is a known set of smooth basis functions, $\boldsymbol{\Delta}_u(z(t))$ is an unmatched uncertainty, and $\boldsymbol{z}(t)$ is the sub-state that contribute to the unmatched uncertainty, and the system matrices $A, \boldsymbol{b}, \boldsymbol{c}^\top, B_u, C_u$ are known. A nominal linear controller:

$$u_{nom}(t) = -\boldsymbol{K}_x^\top \boldsymbol{x}(t) + K_r r(t), \tag{2}$$

is assumed to be designed such that the resulting closed-loop system with the known part of the system in (1), without unmatched uncertainty, satisfies design specifications. Hence we can define a reference model for the desired behavior using

$$\begin{aligned}\dot{\boldsymbol{x}}_m(t) =& A_m \boldsymbol{x}_m(t) + \boldsymbol{b}_m r(t)\\ y_m(t) =& \boldsymbol{c}^\top \boldsymbol{x}_m(t),\end{aligned} \tag{3}$$

where $A_m = A - \boldsymbol{b}\boldsymbol{K}_x^\top$ is Hurwitz, $\boldsymbol{b}_m = \boldsymbol{b}K_r$, and $r(t)$ is a bounded reference command.

Let

$$u(t) = u_{nom}(t) - u_{ad}(t), \tag{4}$$

where $u_{ad}(t)$ is an adaptive signal introduced to approximately cancel the uncertainty $\boldsymbol{W}^\top \boldsymbol{\phi}(\boldsymbol{x}(t))$:

$$u_{ad}(t) = \widehat{\boldsymbol{W}}(t)^\top \boldsymbol{\phi}(\boldsymbol{x}(t)), \tag{5}$$

whose estimate $\widehat{\boldsymbol{W}}(t)$ for the ideal weight $\boldsymbol{W}$ in (1) is updated using:

$$\dot{\widehat{\boldsymbol{W}}}(t) = -\gamma\boldsymbol{\phi}(\boldsymbol{x}(t))\boldsymbol{e}(t)^\top P\boldsymbol{b} - \sigma\widehat{\boldsymbol{W}}(t), \tag{6}$$

where $\gamma > 0 \in \mathbb{R}$ is the adaptation gain, σ is the $\sigma-$modification gain, and $P > 0$ is obtained by solving the following Lyapunov function with a selected $Q > 0$:

$$A_m^\top P + PA_m + Q = 0. \tag{7}$$

Stability results in adaptive control are cast in terms of the tracking error:

$$\boldsymbol{e}(t) = \boldsymbol{x}_m(t) - \boldsymbol{x}(t), \tag{8}$$

whose dynamics are described by:

$$\dot{\boldsymbol{e}}(t) = A_m\boldsymbol{e}(t) + \boldsymbol{b}\widetilde{\boldsymbol{W}}(t)^\top\boldsymbol{\phi}(\boldsymbol{x}(t)) - B_u\boldsymbol{\Delta}_u(\boldsymbol{z}(t)), \tag{9}$$

where $\widetilde{\boldsymbol{W}}(t) = \widehat{\boldsymbol{W}}(t) - \boldsymbol{W}$ is the weight estimation error. From (6), the weight estimation error can be written in terms of $\widetilde{\boldsymbol{W}}(t)$ as

$$\dot{\widetilde{\boldsymbol{W}}}(t) = -\gamma\boldsymbol{\phi}(\boldsymbol{x}(t))\boldsymbol{e}(t)^\top P\boldsymbol{b} - \sigma\widetilde{\boldsymbol{W}}(t) - \sigma\boldsymbol{W}, \tag{10}$$

Let

$$\boldsymbol{\zeta}(t) = [\boldsymbol{e}(t)^\top, \widetilde{\boldsymbol{W}}(t)^\top]^\top. \tag{11}$$

Then the error dynamics composed of the tracking error and the weight estimation error are described by:

$$\begin{aligned}\dot{\boldsymbol{\zeta}}(t) &= \underbrace{\begin{bmatrix} A_m & \boldsymbol{b}\boldsymbol{\phi}(\boldsymbol{x}(t))^\top \\ -\gamma\phi(\boldsymbol{x}(t))\boldsymbol{b}^\top P & -\sigma I_N \end{bmatrix}}_{\bar{A}(\boldsymbol{\rho}(t))}\boldsymbol{\zeta}(t) + \underbrace{\begin{bmatrix} -B_u \\ \mathbf{0} \end{bmatrix}}_{\bar{B}_u}\boldsymbol{\Delta}_u(\boldsymbol{z}(t)) + \underbrace{\begin{bmatrix} 0 \\ -I_N \end{bmatrix}}_{\bar{B}_w}\sigma\boldsymbol{W} \\ \boldsymbol{e}(t) &= \underbrace{\begin{bmatrix} I_n & \mathbf{0}_{n\times N} \end{bmatrix}}_{\bar{C}_e}\boldsymbol{\zeta}(t),\end{aligned} \tag{12}$$

where $\boldsymbol{\rho}(t) = \boldsymbol{\phi}(\boldsymbol{x}(t))$. Following Ref. [13], the matrix $\bar{A}(\boldsymbol{\rho})$ is affinely parametrized as follows. Let Ω_x be a compact domain of interest such that $\boldsymbol{x}(t) \in \Omega_x$ for all $t \geq 0$. Then, the basis function $\boldsymbol{\phi}(\boldsymbol{x}) = [\phi_1(\boldsymbol{x}), \ldots, \phi_N(\boldsymbol{x})]^\top$ is known and hence we can calculate the interval to which each element of the basis function belongs, i.e., $\rho_j = \phi_j(\boldsymbol{x}) \in [\min(\phi_j(\boldsymbol{x})), \max(\phi_j(\boldsymbol{x}))] = [\underline{\phi}_j, \overline{\phi}_j]$. This leads to:

$$\bar{A}(\boldsymbol{\rho}) = A_0 + \sum_{j=1}^{N}\rho_j A_j, \tag{13}$$

where $A_0 = \begin{bmatrix} A_m & \mathbf{0}_{n\times N} \\ \mathbf{0}_{N\times n} & -\sigma I_N \end{bmatrix}$, $A_j \in \mathbb{R}^{(n+N)\times(n+N)}$ is a matrix such that $A_j(1:n, k) = \boldsymbol{b}$, $A_j(k, 1:n) = -\gamma\boldsymbol{b}^\top P$ if $k = j$, and $A_j(k,l) = 0$ otherwise ($k \neq j$ nor $l \neq j$). The notation $i : n$ is used to represent indices from i to n. Note that the affine parameter belongs to the set $\mathcal{P}$ that is given by $\boldsymbol{\rho} \in \mathcal{P} := co(\mathcal{P}_0)$ where

$$\mathcal{P}_0 := \{\boldsymbol{\rho} = (\phi_1, \ldots, \phi_N) \ : \ \phi_j \in \{\underline{\phi}_j, \overline{\phi}_j\}, j = 1, \ldots, N\}. \tag{14}$$

Following the analysis framework in Ref. [14], we address the following problems.

1. What is the size of an unmatched uncertainty that can be tolerated by the control law in (4) in the $\mathcal{L}_2$ input-output stability setting?

2. How much does the performance of the closed loop system degrade given a certain class of static unmatched uncertainties?

In the problem 1, the size is specified by $\mathcal{L}_2$ gain, and therefore a dynamic mapping is allowed. Since it is very difficult to investigate the performance degradation due to dynamic unmatched uncertainties, in the problem 2 we restrict unmatched uncertainties to the class of static mapping whose slopes are upper bounded by a known constant.

III. Analysis for a Tolerable Unmatched Uncertainty

The size of the unmatched uncertainty $\boldsymbol{\Delta}_u$ that does not destroy the stability of the dynamics in (12) is specified in the $\mathcal{L}_2$ input output stability setting.[15] In order to derive a pull-out uncertainty interconnection, we note that $\boldsymbol{z}$ can be, using (8), rewritten as

$$\boldsymbol{z}(t) = \bar{C}_u\boldsymbol{\zeta}(t) + \boldsymbol{z}_0(t), \tag{15}$$

where $\bar{C}_u = [-C_u, 0]$, $\boldsymbol{z}_0(t) = C_u\boldsymbol{x}_m(t)$, and $\boldsymbol{z}_0 \in \mathcal{L}_\infty$. Further, by denoting that $\sigma W = \boldsymbol{w}_0 \in \mathcal{L}_\infty$, $\boldsymbol{w}_u = \boldsymbol{\Delta}_u(\boldsymbol{z})$, the error dynamics in (12) can be rewritten as:

$$\begin{aligned} \dot{\boldsymbol{\zeta}} &= \bar{A}(\boldsymbol{\rho})\boldsymbol{\zeta} + \bar{B}_u\boldsymbol{w}_u + \bar{B}_w\boldsymbol{w}_0 \\ \boldsymbol{z} &= \boldsymbol{z}_1 + \boldsymbol{z}_0, \ \text{where } \boldsymbol{z}_1 = \bar{C}_u\boldsymbol{\zeta} \\ \boldsymbol{w}_u &= \boldsymbol{\Delta}_u(\boldsymbol{z}). \end{aligned} \tag{16}$$

By introducing a time-varying system $\Sigma \sim \begin{bmatrix} \bar{A}(\boldsymbol{\rho}) & \bar{B}_u & \bar{B}_w \\ \bar{C}_u & 0 & \end{bmatrix}$, whose input and output are $(\boldsymbol{w}_u, \boldsymbol{w}_0)$ and $\boldsymbol{z}$, the system in (16) can be depicted by the pull-out interconnection shown in Figure 1. As in Ref. [14], we classify the tolerable size of $\boldsymbol{\Delta}_u$ by its $\mathcal{L}_2$ gain.

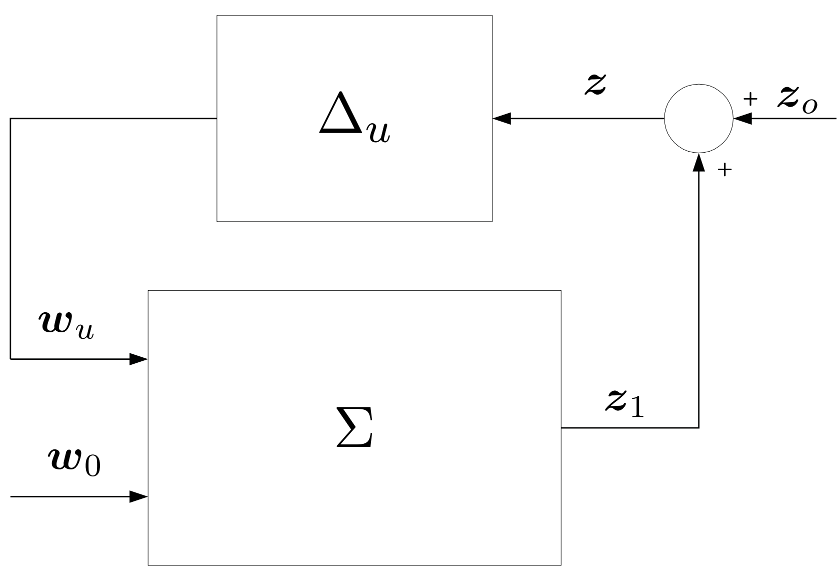


Figure 1. Interconnection with the pull-out of the unmatched uncertainty

Proposition 1. Suppose that we solve the following LMI problem:

$$
\begin{aligned}
&\textit{minimize}\ \gamma_s > 0 \text{ subject to} \\
&\quad X = X^\top > 0 \text{ and} \\
&\begin{bmatrix} \bar{A}(\boldsymbol{\rho})^\top X + X\bar{A}(\boldsymbol{\rho}) + \bar{C}_u^\top \bar{C}_u & X\bar{B}_u \\ \bar{B}_u^\top X & -\gamma_s^2 I \end{bmatrix} < 0,\ \forall \boldsymbol{\rho} \in \mathcal{P}_0.
\end{aligned}
\tag{17}
$$

Then, the closed loop system in (16), depicted in Figure 1, remains stable for all the uncertain mapping $\boldsymbol{\Delta}_u$ whose $\mathcal{L}_2$ gain is less than $1/\gamma_s$.

The proof of Proposition 1 directly follows from the small-gain theorem[16] and the fact that $A(\boldsymbol{\rho})$ is affinely parameterized with respect to $\boldsymbol{\rho}$. Notice that a class of dynamic mapping is allowed as an unmatched uncertainty unlike standard analysis in nonlinear robust control theory.[17] This is because of the fact that the stability problem is pursued using an input-output framework while the standard results are based on application of direct Lyapunov theorem.

IV. Analysis for the Performance Degradation due to a Static Unmatched Uncertainty

Whereas the stability problem in Section III is only concerned with whether all the signals in the closed loop system in (12) remain bounded in the presence of an unmatched uncertainty, the analysis of the degradation in performance due to an unmatched uncertainty requires quantifying the size of disturbance resulting from the unmatched uncertainty. Since specifying the external disturbance of an uncertain dynamic mapping is not straightforward, in this section we restrict our attention to a static unmatched uncertainty whose slope is bounded by a known constant. In other words, we introduce the following assumption.

Assumption 1. The unmatched uncertainty $\boldsymbol{\Delta}_u$ is a static mapping and $\|\boldsymbol{\Delta}_u'(C_u\boldsymbol{x})\| \leq \gamma_u$ for $\forall \boldsymbol{x} \in \Omega_x$, where $\boldsymbol{\Delta}_u' := \frac{d\boldsymbol{\Delta}_u}{d\boldsymbol{z}}$ and $\gamma_u > 0$ is a known constant.

Let

$$
\bar{\boldsymbol{\Delta}}_u(\boldsymbol{\zeta}) := \boldsymbol{\Delta}_u(\boldsymbol{z}) - \boldsymbol{\Delta}_u(\boldsymbol{z}_m), \tag{18}
$$

where $\boldsymbol{z}_m = C_u\boldsymbol{x}_m$. Then by the mean-value theorem[18] together with Assumption 1, we have

$$
\left\|\bar{\boldsymbol{\Delta}}_u(\boldsymbol{\zeta})\right\| \leq \left\|\boldsymbol{\Delta}_u'(\bar{\boldsymbol{z}})(\boldsymbol{z} - \boldsymbol{z}_m)\right\| \leq \gamma_u \left\|C_u\boldsymbol{e}\right\|,
$$

where $\bar{\boldsymbol{z}} = \theta\boldsymbol{z} + (1-\theta)\boldsymbol{z}_m = C_u\bar{\boldsymbol{x}}$, where $\bar{\boldsymbol{x}} = (\theta\boldsymbol{x} + (1-\theta)\boldsymbol{x}_m) \in \Omega_x$ for a $0 \leq \theta \leq 1$. Using the notation $\bar{\boldsymbol{\Delta}}_u$, the closed-loop system in (12) can be rewritten as:

$$
\begin{aligned}
\dot{\boldsymbol{\zeta}} &= \bar{A}(\boldsymbol{\rho})\boldsymbol{\zeta} + \bar{B}_u\bar{\boldsymbol{\Delta}}_u(\boldsymbol{\eta}_u) + \bar{B}_u\boldsymbol{\Delta}_u(C_u\boldsymbol{x}_m) + \bar{B}_w\sigma\boldsymbol{W} \\
\boldsymbol{\eta}_u &= C_u\boldsymbol{e} = \bar{C}_u\boldsymbol{\zeta}, \\
\boldsymbol{\eta}_p &= \boldsymbol{e} = \bar{C}_e\boldsymbol{\zeta}.
\end{aligned}
\tag{19}
$$

Notice that the unmatched uncertainty is now decomposed as the part, $\bar{\boldsymbol{\Delta}}_u(\boldsymbol{\eta}_u)$ with $\bar{\boldsymbol{\Delta}}_u(0) = 0$, that contributes to the stability characteristic of the closed loop system and the part $\boldsymbol{\Delta}_u(C_u\boldsymbol{x}_m)$ that act as an external disturbance . Due to Assumption 1, $\bar{\boldsymbol{\Delta}}_u^\top\bar{\boldsymbol{\Delta}}_u \leq \gamma_u^2\boldsymbol{\eta}_u^\top\boldsymbol{\eta}_u$ for $\forall t \geq 0$. Moreover, by introducing

$$
\bar{B}_p = [\bar{B}_u, \bar{B}_w],\ \boldsymbol{w}_p = [\boldsymbol{\Delta}_u(C_u\boldsymbol{x}_m)^\top, \sigma W^\top]^\top, \boldsymbol{w}_u = \bar{\boldsymbol{\Delta}}_u(\boldsymbol{\eta}_u), \tag{20}
$$

the system in (19) is further written as:

$$\begin{aligned}\dot{\boldsymbol{\zeta}} &= \bar{A}(\boldsymbol{\rho})\boldsymbol{\zeta} + \bar{B}_u\boldsymbol{w}_u + \bar{B}_p\boldsymbol{w}_p \\ \boldsymbol{\eta}_u &= \bar{C}_u\boldsymbol{\zeta},\ \ \boldsymbol{w}_u = \bar{\boldsymbol{\Delta}}_u(\boldsymbol{\eta}_u), \\ \boldsymbol{\eta}_p &= \bar{C}_e\boldsymbol{\zeta}.\end{aligned} \tag{21}$$

With the time-varying system $\Sigma \sim \begin{bmatrix} \bar{A}(\boldsymbol{\rho}) & \bar{B}_u & \bar{B}_p \\ \bar{C}_u & 0 & 0 \\ \bar{C}_e & 0 & 0 \end{bmatrix}$ whose input and output are $(\boldsymbol{w}_u, \boldsymbol{w}_p)$ and $(\boldsymbol{\eta}_u, \boldsymbol{\eta}_p)$, respectively, the system in (21) is depicted by the diagram in Figure 2. Except that Σ

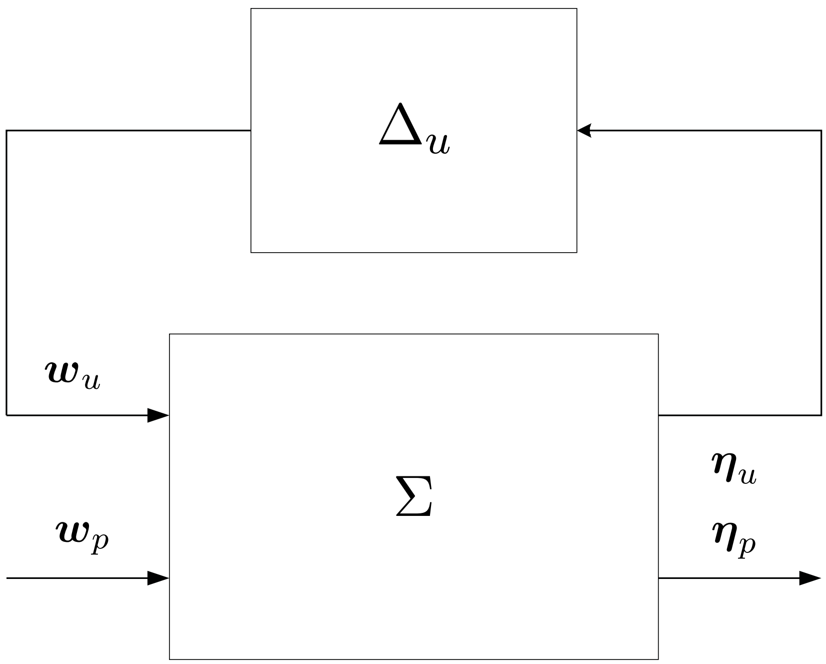


Figure 2. Robust Performance Set-up

is a time-varying system, Figure 2 essentially lays out the diagram of *robust performance*.[19] As a result, a set of LMI tools can be used for performance analysis.

The following convergence analysis is analogous to that in Ref.[13] with the difference being the presence of the unmatched uncertainty $\boldsymbol{\Delta}_u$.

Proposition 2. Suppose that there exists $X = X^\top > 0, \mu > 0$ such that

$$\begin{bmatrix} \bar{A}(\boldsymbol{\rho})^\top X + X\bar{A}(\boldsymbol{\rho}) + \bar{C}_u^\top \bar{C}_u + \mu X & X\bar{B}_u \\ \bar{B}_u^\top X & -\gamma_u^{-2} I \end{bmatrix} < 0,\ \forall \boldsymbol{\rho} \in \mathcal{P}_0. \tag{22}$$

Then $\boldsymbol{\zeta}(t)$ in (21) is exponentially bounded by:

$$\|\boldsymbol{\zeta}(t)\| \le \sqrt{\kappa(X)}\, \|\boldsymbol{\zeta}(0)\|\, e^{-\frac{\mu}{2}t} + 2\sqrt{\kappa(X)}\, \{\|B_u\|\, \|\boldsymbol{\Delta}_u(C_u\boldsymbol{x}_m)\|_\infty + \sigma\, \|\boldsymbol{W}\|\}\, (1 - e^{-\frac{\mu}{2}t}), \tag{23}$$

where $\kappa(X) = \lambda_{\max}(X)/\lambda_{\min}(X)$, $\|\boldsymbol{\Delta}_u(C_u\boldsymbol{x}_m)\|_\infty = \sup_{\tau\in[0,t]} \|\boldsymbol{\Delta}_u(C_u\boldsymbol{x}_m(\tau))\|$.

Proof. Let $\boldsymbol{\zeta}(t) = \Phi(t,0)\boldsymbol{\zeta}(0)$ be a solution to the following system:

$$\dot{\boldsymbol{\zeta}} = \bar{A}(\boldsymbol{\rho})\boldsymbol{\zeta} + \bar{B}_u\bar{\boldsymbol{\Delta}}_u(\bar{C}_u\boldsymbol{\zeta}), \tag{24}$$

where $\Phi(t,0)$ is the transition matrix. Consider $V(\boldsymbol{\zeta}) = \boldsymbol{\zeta}^\top X \boldsymbol{\zeta}$. Multiplying $[\bar{\boldsymbol{\Delta}}_u^\top, \boldsymbol{\zeta}^\top]^\top$ on both sides to Eq.(22) leads to $\dot{V} \leq -\mu V$, $\forall \boldsymbol{\rho} \in \mathcal{P}_0$ because $\bar{\boldsymbol{\Delta}}_u^\top \bar{\boldsymbol{\Delta}}_u \leq \gamma_u^2 \boldsymbol{\eta}_u^\top \boldsymbol{\eta}_u$. Due to affine parametrization of $\boldsymbol{\rho}$, we have $V(t) \leq V(0)e^{-\mu t}$, $\forall \boldsymbol{\rho} \in \mathcal{P}$. From the fact that $\lambda_{\min}(X)\left\|\boldsymbol{\zeta}\right\|^2 \leq V(\boldsymbol{\zeta}) \leq \lambda_{\max}(X)\left\|\boldsymbol{\zeta}\right\|^2$ it follows that $\left\|\boldsymbol{\zeta}(t)\right\|^2 \leq \kappa(X)e^{-\mu t}\left\|\boldsymbol{\zeta}(0)\right\|^2$. Since $\boldsymbol{\zeta}(0)$ is arbitrary,

$$\left\|\Phi(t,0)\right\| \leq \sqrt{\kappa(X)}e^{-\frac{\mu}{2}t}. \tag{25}$$

From this, the solution for the system in (21) is derived as $\boldsymbol{\zeta}(t) = \Phi(t,0)\boldsymbol{\zeta}(0) + \int_0^t \Phi(t,s)\bar{B}_p \boldsymbol{w}_p(t)ds$. Considering that $\left\|\bar{B}_p \boldsymbol{w}_p(s)\right\| \leq \left\|B_u\right\|\left\|\boldsymbol{\Delta}_u(C_u \boldsymbol{x}_m)\right\|_\infty + \sigma\left\|\boldsymbol{W}\right\|$, $\forall s \in [0,t]$ leads to (23). □

Remark 1. The feasibility of LMI in (22)) requires that $\gamma_u \leq 1/\gamma_s$. The guaranteed convergence rate can be found by maximizing μ. In case of the analysis for the nominal convergence rate, the convergence rate obtained by LMI analysis was guaranteed to be less conservative than the standard result in the nominal convergence result.[13] Proposition 2 shows that the LMI analysis for the guaranteed convergence rate can be performed if an upper bound of the slope is known for the unmatched uncertainty. There does not exist any result in standard adaptive control that obtain a guaranteed convergence rate in the presence of an unmatched uncertainty in the literature.

In adaptive control, a primary importance is put on the behavior of the tracking error $\boldsymbol{e}(t)$. Following the same path as that in Ref.[13], the following proposition shows how the tracking error can be analyzed in the presence of the unmatched uncertainty.

Proposition 3. Suppose that there exist $X = X^\top > 0$ and $\mu, \beta, \nu > 0$ such that

$$\begin{gathered}\begin{bmatrix} A(\boldsymbol{\rho})^\top X + XA(\boldsymbol{\rho}) + \mu X + \bar{C}_u^\top \bar{C}_u & X\bar{B}_u & X\bar{B}_p \\ \bar{B}_u^\top X & -\gamma_u^{-2} I & 0 \\ \bar{B}_p^\top X & 0 & -\nu I \end{bmatrix} < 0, \\ \begin{bmatrix} \mu X & 0 & \bar{C}_e^\top \\ 0 & (\beta - \nu)I & 0 \\ \bar{C}_e & 0 & \beta I \end{bmatrix} > 0,\ \forall \boldsymbol{\rho} \in \mathcal{P}_0,\end{gathered} \tag{26}$$

Then the tracking error is upper bounded by:

$$\left\|\boldsymbol{e}(t)\right\| \leq \sqrt{\beta\mu\lambda_{\max}(X)}\left\|\boldsymbol{\zeta}(0)\right\| e^{-\frac{\mu}{2}t} + \beta\left\{\left\|\boldsymbol{\Delta}_u(C_u \boldsymbol{x}_m)\right\|_\infty + \sigma\left\|\boldsymbol{W}\right\|\right\}. \tag{27}$$

Proof. Consider $V(\boldsymbol{\zeta}) = \boldsymbol{\zeta}^\top X \boldsymbol{\zeta}$. By multiplying $[\boldsymbol{\zeta}^\top, \boldsymbol{\Delta}_u^\top, \boldsymbol{w}_p]^\top$ on both sides of Eq. (26), we have $\dot{V} + \mu V - \nu\left\|\boldsymbol{w}_p\right\|_\infty^2 < 0$. This leads to:

$$\begin{aligned} V(t) &\leq V(0)e^{-\mu t} + \nu\left\|\boldsymbol{w}_p\right\|_\infty^2 \int_0^t e^{-\mu(t-s)}ds \\ &\leq V(0)e^{-\mu t} + \nu/\mu\left\|\boldsymbol{w}_p\right\|_\infty^2. \end{aligned} \tag{28}$$

From the second inequality, we have $\left\|\boldsymbol{e}(t)\right\|^2 < \beta[\mu V(t) + (\beta - \nu)\left\|\boldsymbol{w}_p\right\|_\infty^2]$. By substituting (28), we have $\left\|\boldsymbol{e}(t)\right\|^2 < \beta[\mu V(0)e^{-\mu t} + \beta\left\|\boldsymbol{w}_p\right\|_\infty^2] \leq \beta\mu\lambda_{\max}(X)\left\|\boldsymbol{\zeta}(0)\right\|^2 e^{-\mu t} + \beta^2\left\|\boldsymbol{w}_p\right\|_\infty^2$. This leads to (27). □

Compared to the nominal LMIs in Ref.[13], Propositions 2 and 3 shows that the performance degradation due to the unmatched uncertainty occurs from the term γ_u, which affect the dynamic characteristic of the closed loop system, as well as the term $\left\|\boldsymbol{\Delta}_u(C_u \boldsymbol{x}_m\right\|_\infty$, which acts as an additional disturbance to the error dynamics.

V. Simulations

Let us consider the roll dynamics described by:

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= A\boldsymbol{x}(t) + \boldsymbol{b}(u(t) + \boldsymbol{W}^\top \boldsymbol{\phi}(\boldsymbol{x}(t))) + \boldsymbol{b}_m p_{cmd}(t), \\ y(t) &= \boldsymbol{c}^\top \boldsymbol{x}(t) \end{aligned} \tag{29}$$

where the state vector $\boldsymbol{x} = [p_i, v, p, r]$ is composed of an integrator of the roll rate error, the $Y-$axis velocity, the roll rate, and the yaw rate, respectively. The output $y(t)$ represents the roll rate, the control signal $u = \delta_a$ represents the aileron deflection, and $p_{cmd}(t)$ represents the roll rate reference command. The system matrices in (29) are obtained by linearizing the Generic Transportation Model (GTM) model at the angle of attack 2^o trim point and are given by:

$$A = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & -0.8532 & 6.5778 & -186.3175 \\ 0 & -0.8720 & -8.7068 & 1.9306 \\ 0 & 0.3365 & -0.2895 & -2.0953 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 0 \\ -0.0665 \\ -1.7828 \\ -0.0462 \end{bmatrix}^\top, \boldsymbol{c} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}^\top, \boldsymbol{b}_{\mathbf{m}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^\top. \tag{30}$$

The nominal controller in (2) is designed as a linear quadratic regulator (LQR) whose feedback and feedforward gains are:

$$\boldsymbol{K}_x^\top = \begin{bmatrix} 100.0000 & 0.3868 & -6.5963 & -5.3349 \end{bmatrix}, \; K_r = 0. \tag{31}$$

The reference model in (3) is realized as a nominal closed loop system in which the known part of the linear system in (29) is regulated by the LQR controller. The closed-loop system has gain margin (GM) of infinity, the phase margin (PM) of 75.9873 ^{o} at the crossover frequency of 14.7172 rad/s. An analysis for a NN-based adaptive control that augments the LQR controller is presented in Ref. 14.

In this paper, the unmatched uncertainty $\boldsymbol{\Delta}_u$ is derived from a linearized dynamics for a damaged GTM model in which the rudder is off. Moreover, since the absence of the rudder has little effect on the effectiveness of the aileron, it is assumed that the same b is maintained in the absence of the rudder. In other words, the main effect of the rudder damage is assumed to occur in the system A matrix in (30). Linearizing the damaged GTM dynamics at the given trim condition leads to the following system matrix:

$$A_d = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & -0.5235 & 6.659 & -125.7 \\ 0 & -0.5167 & -6.414 & 0.587 \\ 0 & 0.1928 & -0.2571 & -0.9719 \end{bmatrix}. \tag{32}$$

The resulting simulation model employed for our study is described by:

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= (A + \alpha\Delta A)\boldsymbol{x}(t) + \boldsymbol{b}u(t) + \boldsymbol{b}_m p_{cmd}(t), \\ y(t) &= \boldsymbol{c}^\top \boldsymbol{x}(t), \end{aligned} \tag{33}$$

where α is introduced as a scale factor for the unmatched uncertainty, and

$$\Delta A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.3117 & 0.0812 & 60.6175 \\ 0 & 0.3553 & 2.2928 & -1.3436 \\ 0 & -0.1437 & 0.0324 & 1.0874 \end{bmatrix} = B_u^l \Delta A_3 C_u, \tag{34}$$

where

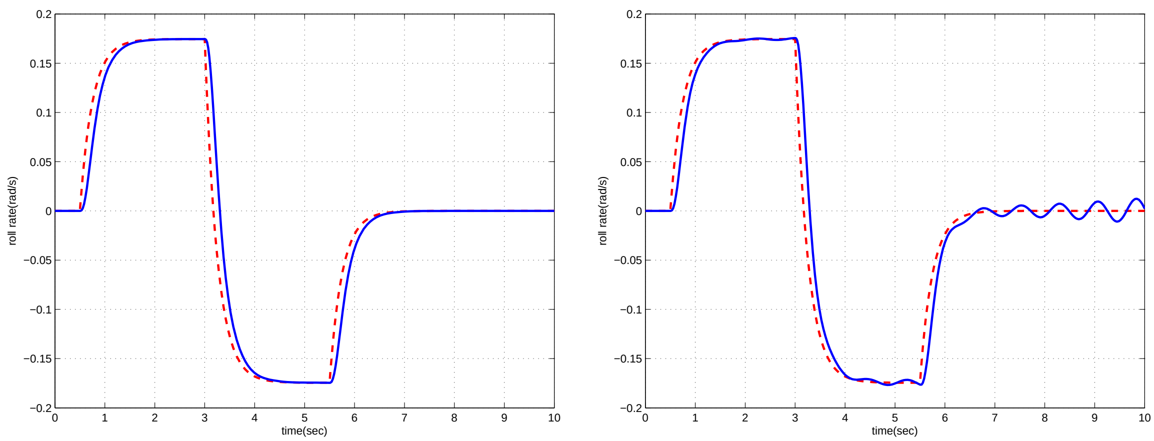
$$B_u^l = diag\{0,1,1,1\}, \Delta A_3 = \Delta A(2:4,2:4), C_u = I_3. \tag{35}$$

1. Linear Control Analysis

Applying the nominal linear controller in (31) to the system in (33) leads to the following closed loop system:

$$\dot{\boldsymbol{x}}(t) = A_m \boldsymbol{x}(t) + \alpha B_u^l \Delta A_3 C_u \boldsymbol{x}(t) + \boldsymbol{b}_m p_{cmd}(t).$$

The analysis of the stability in the presence of norm bounded uncertainty is a standard problem in the literature via the bounded real lemma.[20] Applying the bounded real lemma to the system $\Sigma \sim \begin{bmatrix} A_m & B_u^l \\ C_u & 0 \end{bmatrix}$ leads to the allowable $\mathcal{L}_2$ gain for $\alpha \Delta A_3$ to be 1.8178. Since $\|\Delta A_3\| = 60.6429$, the maximal α for the linear system should be $\alpha = 0.0303$. Figure 3 shows the roll rate tracking responses when there is no unmatched uncertainty ($\alpha = 0$) and when the roll rate response starts to deviate due to the unmatched uncertainty ($\alpha = 1.8$). The roll rate command is dotted in red, and the roll rate response of the system is solid in blue throughout the entire simulation results in this section. It is immediately clear that the performed analysis is quite conservative because the



(a) roll rate tracking with $\alpha = 0$ (b) Roll rate tracking with $\alpha = 1.8$

Figure 3. Time responses with the linear controller

analysis assumes that ΔA_3 is a norm-bounded time-varying matrix while ΔA_3 is actually constant. In other words, the conservatism largely stems from its black-box assumption on the uncertainty $\alpha \Delta A_3$. Note that the same is also true for ensuing analysis for adaptive control because the analysis for an adaptive control essentially follows that of a norm bounded uncertainty. We focus on presenting the LMI framework that is applicable to an adaptive controller in this paper and do not try to perform less conservative LMI analysis. A pursuit for a less conservative LMI approach that incorporates all the structural information available in a specific system is left as a future research topic.

2. Adaptive Control Analysis

In case of adaptive control, notice that the uncertainty $\Delta A \boldsymbol{x}$ can be decomposed as a matched one and an unmatched one via the Gram-Schmidt procedure.[21] That is, the system in (33) can also be written as:

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + \boldsymbol{b}(u + \alpha \boldsymbol{W}^\top \boldsymbol{x}) + B_u \alpha \Delta A_3' C_u \boldsymbol{x}, \tag{36}$$

where $B_u = \begin{bmatrix} \mathbf{0}_{1\times 2} \\ B_3^{\perp} \end{bmatrix}$, $B_3^{\perp} \in \mathbb{R}^{3\times 2}$ is a matrix such that $\boldsymbol{b}_3^{\top} B_3^{\perp} = 0$ and $(B_3^{\perp})^{\top} B_3^{\perp} > 0$,

$$\boldsymbol{W} = \Delta A_3^{\top} \frac{\boldsymbol{b}_3^{\top}}{\boldsymbol{b}_3^{\top}\boldsymbol{b}_3},\ \Delta A_3' = ((B_3^{\perp})^{\top} B_3^{\perp})^{-1} (B_3^{\perp})^{\top} \Delta A_3, \tag{37}$$

and $\boldsymbol{b}_3 = \boldsymbol{b}(2:4)$. According to the system description in (1), this leads to: $\boldsymbol{\phi}(\boldsymbol{x}) = \boldsymbol{x}$, $\boldsymbol{\Delta}_u(\boldsymbol{z}) = \alpha \Delta A_3' \boldsymbol{z}$, and $\boldsymbol{z} = \boldsymbol{x}$. The compact domain of interest is set as $\Omega_x = [-0.1, 0.1] \times [-0.2, 0.2] \times [-0.05, 0.05]$ for (v, p, r).

Tables 1 and 2 show the analysis results when there is no unmatched uncertainty by solving LMIs in Ref. [13]. Very high values in the convergence rate and the small number β indicates that the nominal performance is excellent. For example, the roll rate tracking response with $\gamma = 1000$ and $\sigma = 0.01$ is shown in Figure 4(a).

	$\gamma = 0.1$	$\gamma = 1$	$\gamma = 10$	$\gamma = 10^2$	$\gamma = 10^3$
$\sigma = 0.01$	0.3932×10^6	0.5243×10^6	0.4915×10^6	0.7864×10^6	0.4915×10^6
$\sigma = 0.1$	0.4915×10^6	0.3932×10^6	0.5243×10^6	0.3932×10^6	0.5243×10^6
$\sigma = 1$	0.4588×10^6	3.1457×10^6	0.5243×10^6	0.4915×10^6	0.4915×10^6
$\sigma = 10$	0.4915×10^6	0.4588×10^6	0.4915×10^6	0.3932×10^6	3.1457×10^6
$\sigma = 100$	0.4915×10^6	0.4588×10^6	0.5079×10^6	0.3932×10^6	0.3932×10^6

Table 1. Maximal convergence rate μ in the nominal analysis

	$\gamma = 0.1$	$\gamma = 1$	$\gamma = 10$	$\gamma = 10^2$	$\gamma = 10^3$
$\sigma = 0.01$	0.1467×10^{-3}	0.1322×10^{-3}	0.1243×10^{-3}	0.1721×10^{-3}	0.1663×10^{-3}
$\sigma = 0.1$	0.1131×10^{-3}	0.1719×10^{-3}	0.1667×10^{-3}	0.1392×10^{-3}	0.1811×10^{-3}
$\sigma = 1$	0.1866×10^{-3}	0.1185×10^{-3}	0.0840×10^{-3}	0.1659×10^{-3}	0.1974×10^{-3}
$\sigma = 10$	0.1629×10^{-3}	0.1152×10^{-3}	0.1384×10^{-3}	0.1803×10^{-3}	0.2126×10^{-3}
$\sigma = 100$	0.2202×10^{-3}	0.1978×10^{-3}	0.2208×10^{-3}	0.1982×10^{-3}	0.1947×10^{-3}

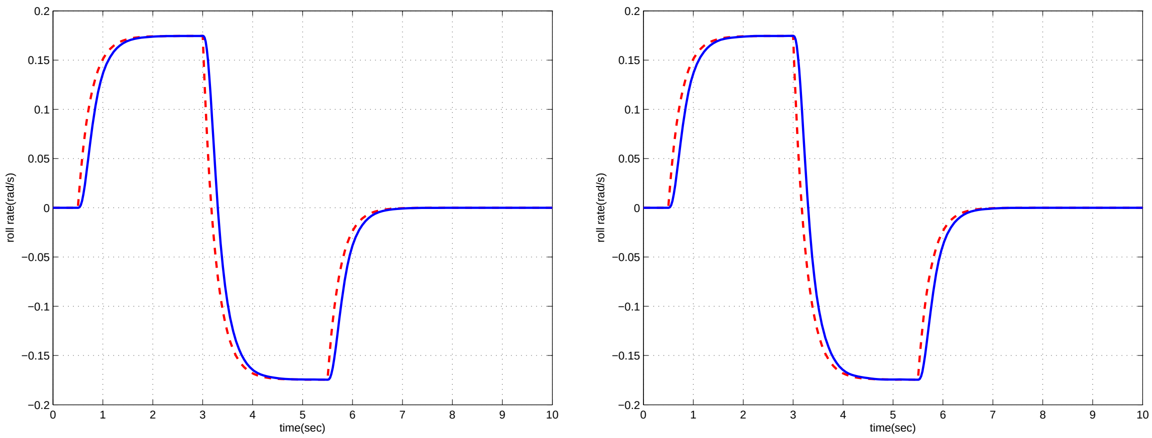
Table 2. UUB parameter β in the nominal analysis

Table 3 shows the allowable $\mathcal{L}_2$ gain for the unmatched uncertainty. Overall within the considered range of the adaptation gain and the $\sigma-$ modification gain, the allowable $\mathcal{L}_2$ gains for the unmatched uncertainty show little variations. Figure 4(b) shows the responses in roll rate tracking

	$\gamma = 0.1$	$\gamma = 1$	$\gamma = 10$	$\gamma = 10^2$	$\gamma = 10^3$
$\sigma = 0.01$	0.1277	0.1276	0.1271	0.1259	0.1025
$\sigma = 0.1$	0.1277	0.1277	0.1276	0.1271	0.1259
$\sigma = 1$	0.1277	0.1277	0.1277	0.1276	0.1271
$\sigma = 10$	0.1277	0.1277	0.1277	0.1277	0.1276
$\sigma = 100$	0.1277	0.1277	0.1277	0.1277	0.1277

Table 3. Tolerable $\mathcal{L}_2$ gain for the unmatched uncertainty

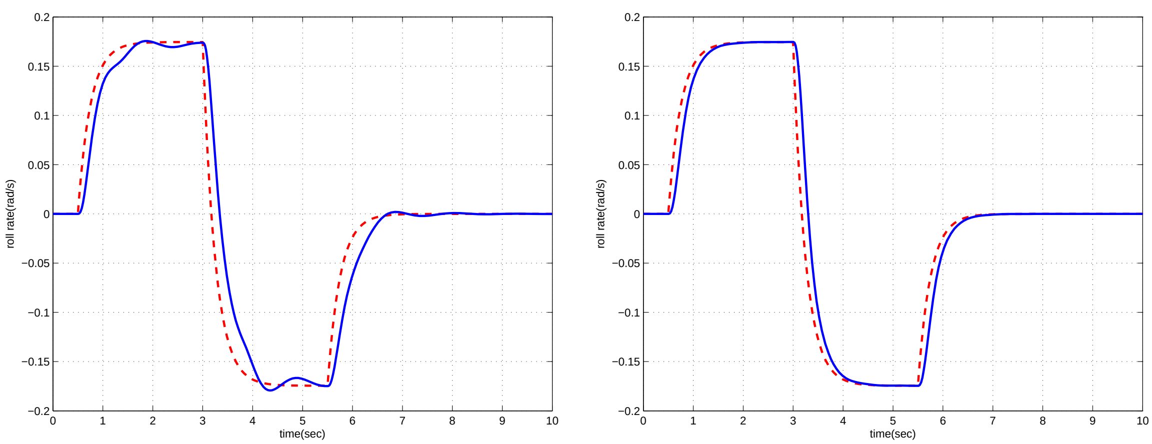
with $\gamma = 1000$ and $\sigma = 0.01$ when $\alpha = 0.13$. There is no change in the tracking performance compared to the roll rate response with $\alpha = 0$ in Figure 4(a). This implies that the $\mathcal{L}_2$ gains in Table 3 is conservative. As a matter of fact, a noticeable performance degradation starts to



(a) roll rate tracking with $\alpha = 0$ (b) Roll rate tracking with $\alpha = 0.13$

Figure 4. Roll rate responses of the adaptive controller with $\gamma = 1000$ and $\sigma = 0.01$.

appear when α is substantially increased. Figure 5(a) shows the roll rate responses when $\alpha = 1.0$. This conservatism can in part be attributed to the black-box assumption on $\alpha \Delta A_3$. Finally, Figure 5(b) shows the roll rate responses when the unmatched uncertainty $\Delta A_3' C_u \boldsymbol{x}$ is replaced by $\Delta A_3(t, \boldsymbol{x}) = [v \sin(10t - \pi/3), -p\cos(4t), r\sin(3t - \pi)]^\top$, which can be tolerated by the adaptive controller due to Proposition 1. No distinguishable performance degradation is observed in roll rate tracking.



(a) roll rate tracking with $\alpha = 1.0$ (b) roll rate tracking with $\alpha = 0.13$ in the presence of a time-varying uncertainty

Figure 5. Investigation of performance degradation of the adaptive controller with $\gamma = 1000$ and $\sigma = 0.01$

Tables 5 and 6 show the parameters associated with analysis of the tracking error. The parameters are obtained by solving LMIs in (26), as the adaptation gain and the $\sigma-$modification gain vary. The slope of the unmatched uncertainty $\boldsymbol{\Delta}_u$ is assumed to be less than 0.05. Compared to the nominal analysis results in Tables 1 and 2, the estimated performance significantly degrades again due to the possible reason that the results in Tables 5 and 6 are guaranteed for any time-varying, static mapping whose slope is bounded by 0.05. In simulations, no significant degradation is observed with the given value of $\alpha = 0.05$. The overall analysis indicates that while a theoretically guarantee can be obtained for the size of an unmatched uncertainty and the performance degradation in

	$\gamma = 0.1$	$\gamma = 1$	$\gamma = 10$	$\gamma = 10^2$	$\gamma = 10^3$
$\sigma = 0.01$	0.0195	0.0195	0.0195	0.0195	0.0195
$\sigma = 0.1$	0.1997	0.1997	0.1997	0.1982	0.1865
$\sigma = 1$	0.9995	1.7847	1.7832	1.7739	1.6978
$\sigma = 10$	1.7847	1.7847	1.7847	1.7847	1.7827
$\sigma = 10^2$	1.3208	0.9976	1.7847	1.7847	1.7847

Table 4. Convergence rate μ in Proposition 2

case of a known upper bound for the unmatched uncertainty, the accuracy of the estimate heavily depends on the structure of an uncertainty that is investigated. Therefore, in order to obtain a less conservative estimate from the LMI analysis, all the structural information should be incorporated in forming LMIs, which is also true for LMI analysis in linear robust control theory.

	$\gamma = 0.1$	$\gamma = 1$	$\gamma = 10$	$\gamma = 10^2$	$\gamma = 10^3$
$\sigma = 0.01$	179.1594	173.5520	157.1020	439.5883	992.8467
$\sigma = 0.1$	75.1509	75.1630	75.2815	77.0400	107.4489
$\sigma = 1$	75.1489	75.1499	75.1597	75.2546	76.3007
$\sigma = 10$	75.1488	75.1489	75.1498	75.1595	75.2525
$\sigma = 10^2$	75.1488	75.1488	75.1489	75.1498	75.1594

Table 5. UUB parameter β in Proposition 3

	$\gamma = 0.1$	$\gamma = 1$	$\gamma = 10$	$\gamma = 10^2$	$\gamma = 10^3$
$\sigma = 0.01$	0.0050	0.0050	0.0150	0.0150	0.0150
$\sigma = 0.1$	0.0450	0.0450	0.0450	0.0550	0.0850
$\sigma = 1$	0.0450	0.0450	0.0450	0.0450	0.0450
$\sigma = 10$	0.0450	0.0450	0.0450	0.0450	0.0450
$\sigma = 10^2$	0.0450	0.0450	0.0450	0.0450	0.0450

Table 6. Corresponding μ in Proposition 3

VI. Conclusions and Future Directions

The framework of LMI-based analysis for adaptive control is extended to a class of systems with an unmatched uncertainty. The formulation involves recasting the error dynamics composed of the tracking error and the weight estimation error into a linear parameter varying form as in the previous approach. We show that the affine parametrization in the previous approach is still valid for quantifying the $\mathcal{L}_{\in}$ gain of a tolerable unmatched uncertainty and that the resulting LMIs resemble a robust stability in the literature. Since an analysis for the performance degradation due to a dynamic unmatched uncertainty is not straightforward, we restrict the performance analysis to the case of a static unmatched uncertainty whose slope is bounded by a known constant. The resulting LMIs resemble a robust performance problem in the literature.

Simulations results with a linzerized damaged GTM model indicate that the LMI analysis results can be quite conservative if an employed assumption is too general for a system at hand as is the case for LMI analysis in linear robust control theory. A simulation study with a less conservative LMI formulation, which incorporates all the available structural information, is left for future research. A stability margin analysis, which essentially follows the path in the authors' previous approach, is also left for future research. It is expected that the unmatched uncertainty introduce an additional parameter, and due to this parameter, the resulting stability margins depend on the slope of the unmatched uncertainties as well. This implies that the stability margin of adaptive control systems depend on both the unknown ideal weight in the matched uncertainty as well as the slope of the unmatched uncertainty.

VII. Acknowledgments

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